The Poor Man's Spherometer

or

A Mechanism for elucidating the Radius of Curvature of a Concave Mirror, as employed in the Astronomical Telescope, utilising the Law of Universal Gravitation, the Laws of Motion derived therefrom, and employing the Method of Fluxions

by

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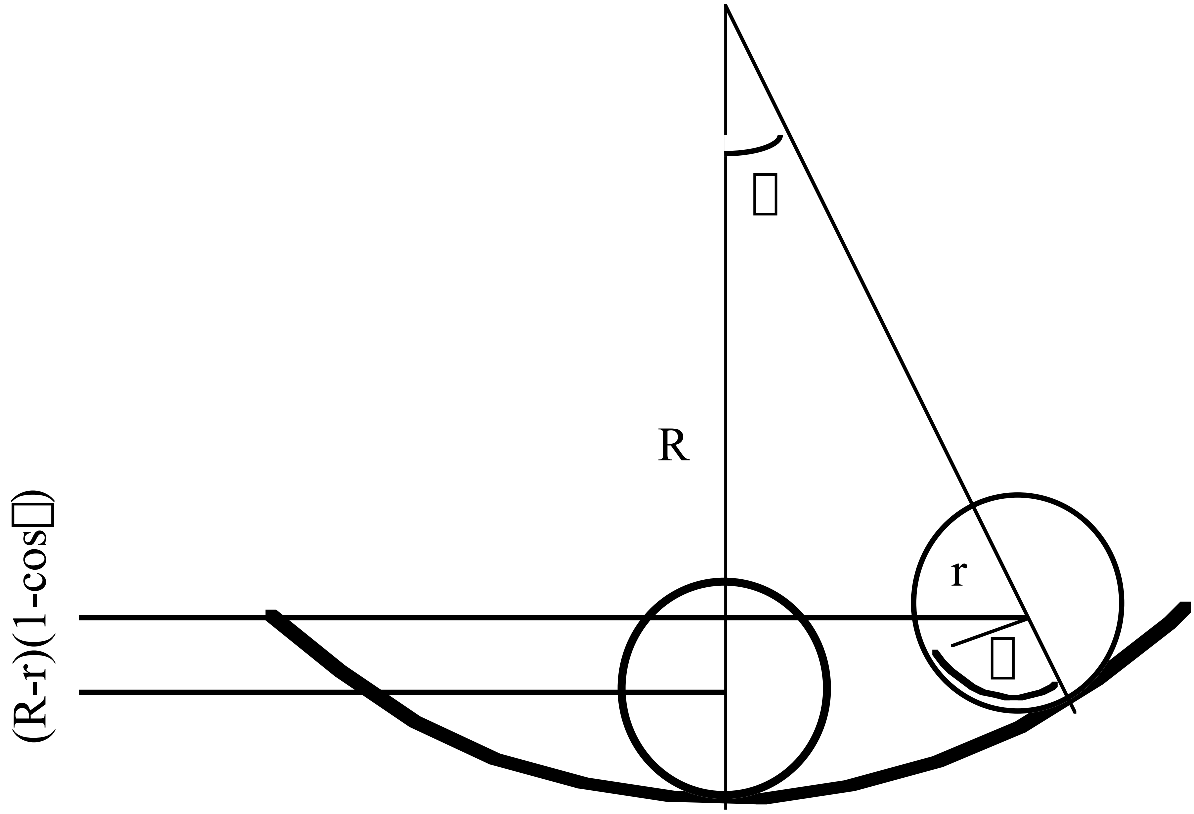
Consider a concave spherical surface of radius R, upon which is rolling without slipping a sphere of mass m and radius r:



The sum of the Kinetic Energy and the Potential Energy in the system must remain constant, that is to say the first derivative of this sum with respect to Time must equal zero. For an homogeneous sphere of mass m and radius r, with Moment of Inertia

$$J_0 = \tfrac{2}{5}mr^2$$

we may write for the Potential Energy

$$PE = mg(R - r)(1 - \cos\theta)$$

and for the two components of Kinetic Energy, translational and rotational, their sum

$$KE = \tfrac{1}{2}m\left[(R - r)\dot{\theta}\right]^2 + \tfrac{1}{2}\left(\tfrac{2}{5}mr^2\right)\left(\frac{R}{r} - 1\right)^2\dot{\theta}^2$$

where the Angular Velocity is $\dot{\theta}$ and where we have substituted for ϕ

$$\phi = \frac{R\theta}{r}$$

Now, the first derivative of the sum of the Kinetic and Potential Energy must be zero, i.e.

$$\tfrac{7}{5}(R - r)^2 m\ddot{\theta} + mg(R - r)\sin\theta = 0$$

where $\ddot{\theta}$ is the Angular Acceleration.

If the angle of oscillation θ is small, then $\sin\theta \approx \theta$; also, if the radius of the sphere r is very small compared with the Radius of Curvature R, then $R - r \approx R$ and we may simplify the above equation:

$$\ddot{\theta} + \frac{5mgR\theta}{7R^2m} = 0$$

and having further simplified by cancellation of m and R, we may deduce that

$$2\pi f = \sqrt{\frac{5g}{7R}} \text{ radians per second}$$

where f is the oscillatory frequency in cycles per second. If instead of the oscillatory frequency f we measure the time τ seconds for one period of oscillation, then inserting the value of 9.81 metres per second, per second for the Acceleration due to Gravity, and allowing that the Focal Length FL is one half of the Radius of Curvature, we obtain:

$$FL = 0.0887\tau^2 \text{ metres, or } FL = 3.49\tau^2 \text{ inches}$$

Hence if the time taken for one complete oscillation of the sphere upon the mirror is five seconds, we deduce that the Focal Length is:

$$FL = 0.0887 \times 5^2 = 2.22 \text{ metres, or } FL = 3.49 \times 5^2 = 87.3 \text{ inches}$$

It is possible to employ a short cylinder in place of the sphere, and this may indeed be preferable, since the sphere has a tendency to move in elliptical orbits around the centre of the mirror instead of travelling always along a diameter. If the cylinder is short and stubby, perhaps one quarter of an inch or five millimetres in length, the error will not be great, but it is necessary to allow for the difference in Moment of Inertia, which for an homogeneous cylinder is given by:

$$J_0 = \tfrac{1}{2}mr^2$$

from whence, proceeding as before:

$$2\pi f = \sqrt{\frac{2g}{3R}}$$

and

$$FL = 0.0828T^2 \text{ metres, or } FL = 3.26T^2 \text{ inches}$$

from which it may be seen that the correction applied against the spherical case is small, a five second oscillation of the cylinder corresponding to a focal length of 2.07 metres or 81.5 inches.

Now, the limitations in this method are these: that no allowance has been made for friction or air resistance, but chiefest of the objections is a practical one; that mirrors of long focal ratio have but a small sagitta from which comes the motive force for propelling the sphere or cylinder, and upon a rough ground glass surface, this energy may be insufficient to secure smooth oscillations of the test sphere or cylinder to and fro upon the mirror surface.

In employing this method, care should be taken that the mirror is level so that the sphere or cylinder naturally rests in the centre of the concavity; however, no great claim of accuracy can be entertained, as it is the merit of this method to provide a quick, simple and above all cheap method for determining the Radius of Curvature and hence the Focal Length of a mirror in the rough ground state, and if it allows the impecunious amateur to approach within ten percent of the intended goal, it will have achieved its intention. Whilst it may be possible to add sophistication, e.g. by use of a stop watch, all that is necessary is a child's marble and a clock possessing a seconds hand. If need be and circumstances permit, more than one oscillation of the test object can be timed and the cumulative time divided by the count of oscillations completed.

N.B. The amateur is cautioned against making this test on a polished or figured mirror, as the rolling friction of the test object will most certainly ruin the fine surface. Finely polished surfaces should be examined only by optical means.